Statistical model with a standard Γ distribution

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(Received 6 February 2004; published 2 July 2004)

We study a statistical model consisting of N basic units which interact with each other by exchanging a physical entity, according to a given microscopic random law, depending on a parameter λ. We focus on the equilibrium or stationary distribution of the entity exchanged and verify through numerical fitting of the simulation data that the final form of the equilibrium distribution is that of a standard Gamma distribution. The model can be interpreted as a simple closed economy in which economic agents trade money and a saving criterion is fixed by the saving propensity λ. Alternatively, from the nature of the equilibrium distribution, we show that the model can also be interpreted as a perfect gas at an effective temperature $T(\lambda)$, where particles exchange energy in a space with an effective dimension $D(\lambda)$.

DOI: 10.1103/PhysRevE.70.016104 PACS number(s): 89.65.Gh, 87.23.Ge, 02.50.−r

I. INTRODUCTION

Statistical physicists like to understand how and why systems evolve from an initial state toward an equilibrium macroscopic state. The equilibrium state, far from being just the "final state" of the dynamical evolution, actually reflects the details of the underlying dynamics. To write down the "microscopic equation" governing the dynamics of the evolution is a major goal. The various probability distributions, resulting from the different corresponding microscopic equations, have a relevant interest, in that they can be used to derive most of the macroscopic properties of the system. One of the foremost examples is the Maxwell-Boltzmann distribution for the velocities, which can be obtained as a solution of the equation which Boltzmann proposed for the evolution of the probability distribution for a dilute gas.

One of the current challenges is to write down the microscopic equation which would correspond to the century old Pareto law [1] in Economics, stating that the higher end of the distribution of income $f(x)$ follows a power-law

$$f(x) \propto x^{-1-\alpha},$$

where x is the income (money) and the exponent α has a value in the interval 1–2 [2–5]. To this aim, several studies have been made to investigate the characteristics of the real income distribution and provide theoretical models or explanations. For example, Levy and Solomon studied the generalized Lotka-Volterra equations in relation to power-law wealth distribution [6,7], whereas Ispolatov *et al.* [8] studied random exchange models of wealth distributions. Other related studies of exchange models in closed economies have followed [9–14] and, recently, some different approaches have been used to study wealth distributions [15–18]. In general, from these studies it emerges that it is possible to obtain power law distributions in the framework of some economy models, whereas other models predict exponential tails of the income distribution. However, an understanding of the dependence of the distributions on the underlying mechanisms and parameters is still missing. For this reason, it is our general aim to study a statistical model of closed economy, which can be either solved exactly or simulated numerically, and analyze the relation between the microscopic equation and the kind of macroscopic money distribution it results in. This study can be of particular interest, since it can provide some insight as to under what conditions the Pareto law arises.

In this paper, we study a statistical model consisting of N basic units which interact with each other by exchanging a physical entity x, according to a given microscopic law with one constant parameter λ. We study the stationary probability distributions $f(x)$ for different values of λ. Furthermore, we verify through numerical studies that the final form of the equilibrium distribution $f(x)$ is that of a standard Gamma distribution. Then in Sec. II, we interpret the model as a simple closed economy in which agents trade money and have a saving criterion fixed by the saving propensity λ. In Sec. III, using the nature of the equilibrium distribution, we show that the model can also be interpreted as a perfect gas at an effective temperature $T(\lambda)$, made up of particles exchanging energy in a space with an effective dimension $D(\lambda)$. Finally, in Sec. IV, we draw conclusions.

II. THE MODEL ECONOMY

We begin by considering a simple model of closed economy, in which N agents can exchange money in pairs between themselves. All the agents can be initially assigned the same money amount $\overline{x}$, since this condition is not restrictive. Agents are then let to interact and, at every "time step," a pair (i,j) is randomly chosen and the transaction carried out. During the transaction, the agent money amounts x_i and x_j undergo a variation, in which they are randomly reas-

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signed between the two agents—with or without "savings" criterion. We are aware that modeling economy systems by agents exchanging money randomly sounds unrealistic, but, as it appears clearly in the following, the specific form of the microscopic law is not essential for the issues dealt with in the paper. Rather, the main point here is its conservative character. The exchange law is such that the money is conserved during the transaction, i.e., $x_i+x_j=x_i'+x_j'$, where x_i' and x_j' are the money values after the transaction. This implies that at any time the average initial money $\overline{x}$ also represents the average money, $\langle x\rangle\equiv\overline{x}$. The generalized equations which describe the earlier transactions are

$$x_i' = \lambda x_i + \epsilon(1-\lambda)(x_i+x_j),$$

$$x_j' = \lambda x_j + (1-\epsilon)(1-\lambda)(x_i+x_j), \qquad (1)$$

where λ is called the "saving propensity" and ϵ is a random number uniformly distributed in the interval $(0,1)$.

A. Case with $\lambda=0$

First, we deal with the case where there is no saving criterion and $\lambda=0$ in Eqs. (1). The exchanges are then made according to the following law:

$$x_i' = \epsilon(x_i+x_j),$$

$$x_j' = (1-\epsilon)(x_i+x_j). \qquad (2)$$

It can be noticed that, in this model, agents have no debts after the transaction, i.e., they are always left with a money amount $x\geqslant 0$ or, equivalently, that x_i is a positive definite quantity, if $\overline{x}>0$.

It can be shown that, as a consequence of the conservation of money, the system relaxes toward a Gibbs money distribution [9–11]:

$$f_1(x) = \frac{1}{\langle x\rangle}\exp\left(-\frac{x}{\langle x\rangle}\right), \qquad (3)$$

where $\langle x\rangle$ represents the average money. This means that, after the relaxation, most of the agents have a very small amount of money, while the number of very rich agents is exponentially small. In other words, for a given $x'>0$, the number of agents with $x>x'$, as well as the total amount of money they own, exponentially decreases with x'. The equilibrium state represented by the Gibbs distribution (3) has been shown to be robust, in that it is reached independently of the initial conditions and also in models with multiagent transactions.

We have reobtained the exact Gibbs solution for the case $\lambda=0$ by numerical simulations of a system with $N=500$ agents, each agent having initially a money amount $\overline{x}=1$. The system was evolved for 10^6 time steps—i.e., transactions—in order to reach equilibrium, and the final equilibrium distributions were averaged over 10^5 different runs. Figure 1 shows that the numerical results (open circles for $\lambda=0$) are in good agreement with the Gibbs distribution (continuous line).

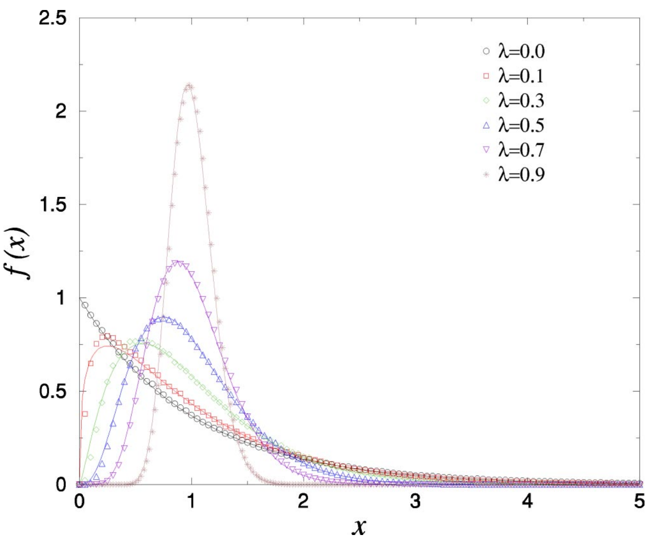

FIG. 1. Equilibrium money distributions for different values of the saving propensity λ, in the closed economy model defined by Eqs. (2) and (4). The continuous curves are the fitting functions, defined in Eq. (6) with the values of n governed by Eq. (5). Note that for the simulation $\langle x\rangle=1$.

B. Case with $\lambda>0$

We now deal with the case where there is a saving criterion, by assuming that the saving propensity, which represents the fraction of money saved before carrying out the transaction, is nonzero, i.e., $\lambda>0$ [9,11]. Conservation of money still holds, $x_i+x_j=x_i'+x_j'$, but the money which can be reassigned in a transaction between the ith and the jth agent has now decreased by a factor $(1-\lambda)$. The exchanges are made according to Eqs. (1), which can also be rewritten as follows:

$$x_i' = x_i + \Delta x,$$

$$x_j' = x_j - \Delta x,$$

$$\Delta x = (1-\lambda)[\epsilon x_j - (1-\epsilon)x_i], \qquad (4)$$

in which money conservation is manifest.

We studied the equilibrium distribution of this model through numerical simulations, for various values of λ, for $N=500$ agents, again each agent having money $\overline{x}=1$ in the initial state. In each simulation a sufficient number of transactions, as far as 10^7, depending on the value of λ, was used in order to reach equilibrium. The final equilibrium distributions, for a given λ, were obtained by averaging over 10^5 different runs. The numerical data are shown in Fig. 1 (cases $\lambda\neq 0$).

We found an analytic form for the equilibrium distribution, for a given λ $(0<\lambda<1)$, which turns out to fit extremely well all data [19]. The function is conveniently expressed in terms of the parameter

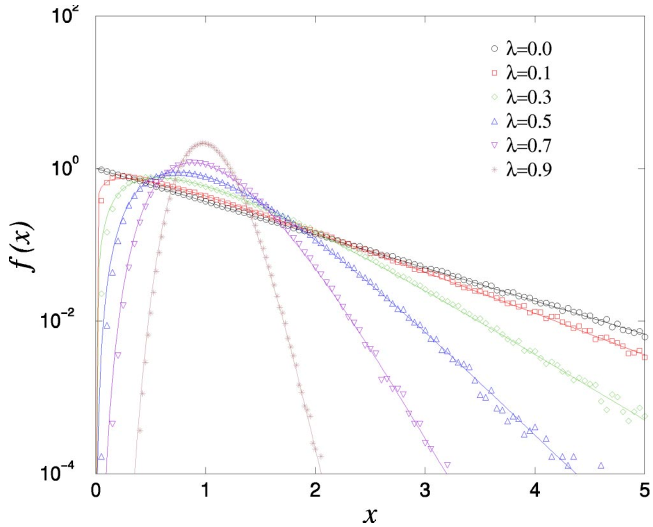

FIG. 2. Same quantities as in Fig. 1, but on a linear-logarithmic scale. Note that for the simulation $\langle x\rangle = 1$.

$$n(\lambda) = 1 + \frac{3\lambda}{1-\lambda}. \tag{5}$$

This particular form of $n(\lambda)$ was suggested by a mechanical analogy, discussed in Sec. III, between the closed economy model with N agents and the dynamics of a gas of N interacting particles. Then the money distributions, for arbitrary values of λ, are well fitted by the function

$$f_n(x) = a_n\, x^{\,n-1} \exp(-\, nx/\langle x\rangle),$$

$$a_n = \frac{1}{\Gamma(n)}\left(\frac{n}{\langle x\rangle}\right)^n, \tag{6}$$

where n is defined in Eq. (5) and the prefactor a_n, where $\Gamma(n)$ is the Gamma function, is fixed by the normalization condition $\int_0^\infty dx\, f_n(x) = 1$.

The fitting curves for the distribution (continuous curves) are compared with the numerical data in Fig. 1. The fitting describes the distribution also for small values of $f(x)$, as shown by the linear-logarithmic plots in Fig. 2. In Fig. 3, the numerical values of the parameters $n(\lambda)$ and $a_n(\lambda)$ obtained directly from fitting the data (shown as dots) are compared with the respective fitting functions (5) and (6) (shown as continuous curves).

By introducing the rescaled variable

$$\xi = nx/\langle x\rangle, \tag{7}$$

the probability distribution (6) can be rewritten as

$$\frac{\langle x\rangle}{n} f_n(x) = \frac{1}{\Gamma(n)} \xi^{\,n-1} \exp(-\,\xi) \equiv \gamma_n(\xi), \tag{8}$$

where $\gamma_n(\xi)$ is the standard Gamma distribution [20,21]. The cumulative distribution for $\gamma_n(\xi)$ is the incomplete Gamma function $\Gamma(\xi, n) = \int_\xi^\infty d\xi'\ \gamma_n(\xi')$:

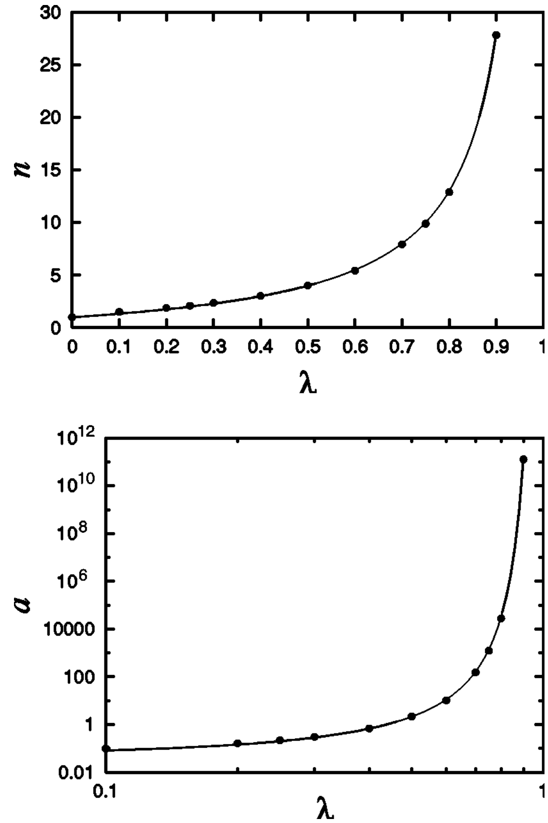

FIG. 3. The parameters n (top) and a_n (bottom) vs λ, obtained from numerical data (dots) and the corresponding analytical formulas (continuous curves) given by Eqs. (5) and (6), respectively. Note that for the simulation $\langle x\rangle = 1$.

$$\gamma_n(\xi) \equiv -\frac{d}{d\xi}\frac{\Gamma(\xi,n)}{\Gamma(n)}. \tag{9}$$

The Gibbs distribution (3) is a special case for $n=1$. The term $\langle x\rangle/n$, on the left-hand side of Eq. (8), is just the scaling factor appearing when the change of variable, from ξ to x, is made in the last equation in order to obtain the distribution $f_n(x)$ for the variable x.

We notice that, with respect to the Gibbs distribution (3), the distribution defined by Eqs. (6) contains the power x^{n-1} and the factor n in the exponential, which qualitatively change the distribution shape. First, they lead to a mode, x_m, different from zero. The mode is shown as a function of the parameter λ in Fig. 4, where the dashed curve represents the theoretical prediction that the mode $x_m = 3\lambda/(1+2\lambda)$ obtainable from Eq. (6). Second, the presence of the factor n is relevant for the mechanical analogy, considered in detail in Sec. III. Finally, in the limit $\lambda \to 1$ (i.e., $n \to \infty$), the distribution $f_n(x)$ tends to a Dirac δ function, peaked around the average value $\langle x\rangle$. A qualitative picture of the evolution of the shapes of the $f_n(x)$'s, for λ going from zero to unity, is obtained by inspection of the various curves in Fig. 1. A more rigorous derivation of the asymptotic distribution for $\lambda \to 1$ can be made by studying the characteristic function $\phi(q)$ [20,21]. The Gamma distribution $\gamma_1(\xi)$ for the dimensionless variable ξ has a characteristic function $\phi_1(q)$

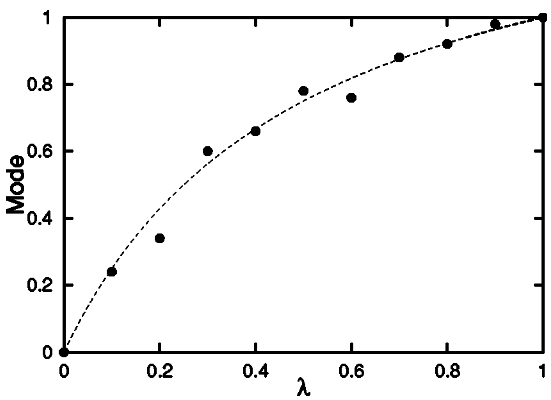

FIG. 4. Variation of the mode with the parameter λ. Dots represent the numerical data, while the dashed curve was obtained theoretically.

$=\int_0^{+\infty} d\xi \exp(iq\xi)\gamma_1(\xi)=1/(1-iq)$. The characteristic function of the Gibbs distribution $f_1(x)$ in Eq. (3) is obtained by rescaling q by the constant factor $x/\xi=\langle x\rangle$:

$$\phi_1(q) = (1 - iq\langle x\rangle)^{-1}. \tag{10}$$

The characteristic function of the generic Gamma distribution $\gamma_n(\xi)$ is simply given by the nth power of $\phi_1(q)$ [20,21], $\phi_n(q)=1/(1-iq)^n$. Analogously, the corresponding characteristic function of $f_n(x)$, Eq. (6), is obtained by scaling q by $x/\xi=\langle x\rangle/n$:

$$\phi_n(q) = (1 - iq\langle x\rangle/n)^{-n}. \tag{11}$$

Thus, in the limit $n\to\infty$ $(\lambda\to 1)$, one obtains

$$\phi_n(q) \to \exp(iq\langle x\rangle). \tag{12}$$

The corresponding distribution is obtained by transforming back the characteristic function, i.e.,

$$f_n(x) = (2\pi)^{-1}\int_{-\infty}^{+\infty} dq\ \exp(-iqx)\phi_n(q) \to \delta(x - \langle x\rangle). \tag{13}$$

This limit shows that a large saving criterion leads to a final state in which economic agents tend to have the same amount of money and, in the limit of $\lambda\to 1$, they all have the same amount $\langle x\rangle$.

III. THE GAS MODEL

The equilibrium distributions (3) can also be interpreted as the Gibbs distribution of the energy x, for a gas at temperature $T=\langle x\rangle/k_B$. This establishes a link between the type of closed economy models considered here and statistical systems, suggesting a re-interpretation of the economy model in terms of a mechanical system of interacting particles. The introduction of a saving parameter $\lambda>0$ changes the shape of the Gibbs distribution into that of a Gamma distribution, but the correspondence with a mechanical system is lost only apparently. In fact, the Gibbs distribution (3) can represent the distribution of kinetic energy x only in $D=$ two dimensions, when its average value is given by $\langle x\rangle_2=2(k_BT/2)$. In all other cases ($D\neq 2$), it is easy to show, starting from the Maxwell-Boltzmann distribution for the velocity in D dimensions, that the equilibrium kinetic energy distribution $f(x)$ coincides, apart from a normalization factor, with the Gamma-distribution $\gamma_n(\xi)$ with $n=D/2$ for the reduced variable $\xi=Dx/2\langle x\rangle_D$:

$$f(x) = \frac{\left(\frac{D}{2\langle x\rangle_D}\right)^{D/2}}{\Gamma\left(\frac{D}{2}\right)} x^{D/2-1} \exp\left(-\frac{Dx}{2\langle x\rangle_D}\right),$$

$$\langle x\rangle_D = D\langle x\rangle_1 = \frac{Dk_BT}{2}, \tag{14}$$

where $\langle x\rangle_D$ represents the average value of kinetic energy in D dimensions. The analogy between the factor D in the argument of the exponential function in Eq. (14) and the analogous factor n in Eq. (6) is to be noticed. The main difference is that, while D is an integer number by hypothesis, the parameter $n(\lambda)$ can assume in general any real values larger than or equal to one.

In Eq. (14) temperature appears implicitly as $T=2\langle x\rangle_D/k_BD$. This suggests that also in the closed economy model considered above the effective temperature of the system should be defined as $\langle x\rangle/n$, rather than $\langle x\rangle$. This is a natural consequence of the fact that the average value of kinetic energy in D dimensions is proportional to D, due to the equipartition theorem, and that an estimate of the amplitude of thermal fluctuations, which is independent of its effective dimension, can be obtained from the ratio $\langle x\rangle_D/D$.

Direct comparison between Eqs. (14) and (6) leads to a formal but exact analogy, between money in the closed economy model considered earlier, with N agents, saving propensity $0\leq\lambda\leq 1$, and given average money $\langle x\rangle$, on one hand, and kinetic energy in an ensemble of N particles in D dimensions at temperature T, on the other, if the effective dimension and temperature are defined as

$$D(\lambda) = 2\ n(\lambda) = \frac{2(1+2\lambda)}{1-\lambda},$$

$$T(\lambda) = \frac{\langle x\rangle}{n(\lambda)} = \langle x\rangle\frac{1-\lambda}{1+2\lambda}, \tag{15}$$

respectively. This equivalence can be qualitatively understood in terms of the underlying microscopic dynamics by considering the example of a fluid of interacting particles. In one dimension, particles undergo head-on collisions, in which they can exchange the total amount of energy they have. In an arbitary (large) number of dimensions, however, this is not possible for purely kinematic reasons and only a fraction of the total energy is actually released or gained on average in a collision. Since the equipartition theorem implies that on average kinetic energy is equally shared among the D dimensions, one can expect that, during a collision, only a fraction $\sim 1/D$ of the total energy is exchanged (and that a corresponding fraction $\lambda \sim 1-1/D$ is "saved"). This estimate $\sim 1/D$ of the exchanged energy is to be compared with the expression for the fraction of exchanged money obtained from Eq. (5) using $n=D/2$, namely $1-\lambda=3/(D/2+2)$, which was in fact found starting the fitting of the numerical data from a function prototype of a form similar to $1/D$.

IV. CONCLUSIONS

We have studied a statistical model, which can be interpreted as a generalization of the simple closed economy model, in which a random reassignment of the total agent money x, involved in the transaction, takes place. The generalized model is characterized by N agents carrying out transactions according to a saving criterion, determined quantitatively through a saving propensity $\lambda > 0$. Alternatively, it can be considered as representing a gas of N interacting particles which on average exchange only a fraction of their kinetic energy x, during a collision. We have shown the existence of such an analogy, by empirically obtaining the corresponding analytical solution $f_n(x)$ for the equilibrium distribution from numerical data.

In both cases the equilibrium distribution can be written as a Gamma distribution $\gamma_n(\xi)$, where the reduced variable is given by $\xi=nx/\langle x\rangle_n$ and $\langle x\rangle_n$ represents the average value of x, given by $\langle x\rangle_n=n\langle x\rangle_1$. The equivalence is represented by $n=n(\lambda)=1+3\lambda/(1-\lambda)$ being a function of the saving propensity on one hand ($\langle x\rangle_1$ is the average value for $\lambda=0$), and by $n=D/2$ being just the half of the number of dimensions, on the other hand ($\langle x\rangle_1$ being in this case the average value in two dimensions).

The fact that we obtain basically the same equilibrium distribution, characterizing the kinetic energy of a gas of particles, suggests some general considerations about closed-economy models. The mechanical analogy illustrated earlier can be addressed to the fact that the system is described statistically by a microcanonical ensemble, just as a closed mechanical system, in which the exchanged quantity is conserved. Thus, a saving propensity larger than zero or any other change in the microscopic law can be expected to lead to a different shape of the equilibrium distribution, as shown in the present work, in which, e.g., the effective number of dimensions and temperature may be different. However, the simple fact that the money is conserved implies that one cannot obtain an arbitrary distribution but, rather, only equilibrium distributions related directly to the microcanonical ensemble.

ACKNOWLEDGMENTS

The authors are thankful to A. Chatterjee, S. Redner, and F. Slanina for useful correspondence and comments. M.P. is grateful to P. Muratore-Ginanneschi for useful discussions. A.C. would like to acknowledge A. Mukherjee for correcting an important mistake in an earlier version. This work was partially supported by the Academy of Finland, Research Center for Computational Science and Engineering, Project No. 44897 (Finnish Centre for Excellence Program 2000–2005). The work at Brookhaven National Laboratory was carried out under Contract No. DE-AC02-98CH10886, Division of Material Science, U.S. Department of Energy.

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